ABSTRACT
Public blockchains are state machines replicated via distributed consensus protocols. Information on blockchains is public by default—marking privacy as one of the key challenges.

We identify two shortcomings of existing approaches to building blockchains for general privacy-preserving applications, namely (1) the reliance on external trust assumptions and (2) the dependency on execution environments (on-chain, off-chain, zero-knowledge, etc.) with heterogeneous programming frameworks.

Towards solving these problems, we propose PESCA—a privacy-enhancing smart contract architecture. PESCA utilizes generic building blocks such as threshold fully-homomorphic encryption (FHE), distributed key generation (DKG), dynamic proactive secret sharing (DPSS), Byzantine-fault-tolerant (BFT) consensus, and universal succinct non-interactive zero-knowledge proofs (zk-SNARKs).

First, we formalize the problem of replicating state machines augmented with threshold decryption protocols and discuss how existing BFT consensus protocols can be adapted to this setting. We describe how to instantiate a blockchain with a fixed FHE public key and have FHE-encrypted chain states programmatically decrypted via consensus.

Next, we describe a smart-contract framework for engineering privacy-preserving applications, where programs are expressed—in a unified manner—between four types of computation: transparent on-chain, confidential (FHE) on-chain, user off-chain, and zero-knowledge off-chain.

Lastly, to showcase the generality and expressiveness of PESCA, we provide two simple application designs for constant function via consensus.

1 INTRODUCTION
Public blockchains, a.k.a. distributed ledgers, are publicly-accessible state machines replicated via distributed consensus protocols. Public blockchains allow anyone to run a consensus node and any user to submit transactions to be processed by the state machine. As a result, most blockchain applications do not offer any privacy guarantee, leaving the financial information of users to public scrutiny.

There has been three main approaches to building general privacy-preserving applications in the public blockchain setting. The first is employing trusted execution environments (TEEs), such as Intel SGX. This is pioneered in the work of Ekiden [19] and implemented in projects such as Oasis and Secret Network. Such an approach effectively replaces the trust on the decentralized node operators to the, very much centralized, hardware providers.

The second approach is delegation of computation to off-chain trusted or multi-party-computation-based (MPC) services, following the pioneering work of Hawk [43]. There have been efforts on reducing the trust required on managers using zero-knowledge and MPC techniques [6]. But solutions in this space remain reliant on trust assumptions that are external to the blockchain in question.

Finally, an approach that has seen a huge surge in interest is use of non-interactive zero-knowledge proofs (NIZKs, or more colloquially ZKPs), following the pioneering work of Zerocash [8] and recent advances in succinct proofs systems (zk-SNARKs) such as Groth16 [39], Marlin [20], and Plonk [34]. There are many known techniques to obtain private direct payments on public blockchains using ZKPs [8, 15, 29, 33]. There have also been efforts on extending the approach of ZKPs to general programs. The most notable effort is the work of Zexe [12], which allows any general program to be executed off-chain and verified on-chain against state commitments in zero-knowledge. However, general zero-knowledge state updates cannot be made concurrent [25]. As a result, shared-state applications on Zexe-like platforms require usage and trust (in terms of privacy) of off-chain sequencers.

Summarizing the state of affairs, there are no known designs of blockchains that natively support programming of general privacy-preserving applications without external trust assumptions.

Problem 1: Can we build blockchains supporting general privacy-preserving applications without external trust assumptions?

Smart contracts and privacy-preserving applications. Smart contract architectures are the “operating systems” for blockchains, enabling user-programmable applications that time-share the execution capabilities of blockchains. However, for privacy preserving applications, there are three distinct execution environments that each application has to program for: on-chain, off-chain (done by user), and zero-knowledge computation. For example, the Ethereum blockchain uses the Ethereum virtual machine (EVM) execution environment, where code is often written in a contract programming language called Solidity. Client-side, in-browser, compute require frameworks such as web3.js [4] and is written in Javascript. Zero-knowledge circuits are programmed with libraries such as Circom [3] and Zokrates [5] in custom domain-specific languages. The heterogeneous nature of these tools makes it harder to design and engineer privacy-preserving applications.

Problem 2: Can we build smart contract platforms where privacy-preserving applications can be designed and engineered in a unified framework?

PESCA: Privacy-Enhancing Smart-Contract Architecture. Towards solving the above two key problems, we propose PESCA—a blueprint of a blockchain and a unified smart contract architecture to build privacy-preserving applications relying solely on the honest super majority—or so-called Byzantine—assumption of consensus nodes.

We provide a technical overview in the rest of the section.

Threshold FHE and key management (Section 2). The high-level idea is simple, we utilize fully homomorphic encryption (FHE) to
compute over encrypted program states and inputs. To not have any single authority over these encrypted values, we utilize threshold cryptography to entrust these encrypted values to the chain, i.e. the consensus set. Specifically, we utilize a threshold FHE scheme [11] supporting Shamir-type keys [50], as well as are well-studied mechanisms to generate and maintain Shamir secret shares among a dynamic consensus sets, via the use of distributed key generation (DKG) [26, 40] and dynamic proactive secret sharing (DPSS) [37, 40, 45] schemes.

Blockchain protocols supporting threshold decryption (Section 3). There has been a vast amount of research on Byzantine fault tolerance (BFT) distributed consensus and state machine replication. The canonical setting requires a state machine with an efficiently computable update function as it is computed by all consensus nodes. In contrast, state machines of interest in our work are not efficiently computable by any single party, due to our usage of FHE threshold decryption. Instead, augmenting the efficient state update is a threshold protocol between consensus nodes. We first formalize and study the replication of state machines with such threshold release protocols.

A $(t, n)$-threshold release protocol is a one-round broadcast protocol between $n$ parties that achieves desired security properties assuming $t$ nodes are honest. Roughly, security asks that the release result $\text{Release}(st)$ to be uncomputable by any adversary unless some honest node releases its share $\text{PartRelease}(sk_i, st)$, even with $t + 1$ adversarially controlled nodes.

Next, we study the replication of state machines with threshold release protocols where the threshold $t$ is set to the BFT-type guarantee of $2f + 1$, where $f$ is the number of faulty or adversarial nodes. We make a simple observation that any BFT-type protocol (e.g. [14, 18, 51]) satisfying safety and liveness can be modified to replicate threshold release state-machines satisfying new notions called release safety and release liveness (assuming partially synchrony).

At this point, we are ready to list the main ingredients of our blockchain. We will fix:

- A $(t, n)$-threshold FHE scheme FHE with Shamir secrets, where $t$ is between $f + 1$ and $2f + 1$. We assume that there is a distributed key generation phase where a chain FHE key $pk$ is computed (and fixed for the lifetime of the chain) and each consensus node obtains its FHE secret key $sk_i$.
- A state machine $M$ (whose exact functionality is fixed later) where an explicitly marked component of each state contains FHE ciphertexts to be decrypted and fed back to the machine as part of subsequent inputs. Note that $M$ is a state machine with a threshold release protocol.
- A BFT-type consensus protocol satisfying liveness and correctness to replicate $M$.
- A universal (or transparent) zk-SNARK system $\Pi$.

We are interested in programming the above state machine $M$ in an expressive manner utilizing the zk-SNARK.

The Smart Contract Architecture (Section 4). We design a smart-contract architecture enabling expressive programming of privacy-preserving applications. We do so by homogenizing different forms of computation. In particular, we consider four types of computation supported by Pesca and required by privacy-preserving applications:

- **Transparent on-chain computation** is the bread and butter of smart contracts. They are replicated among all consensus nodes in the plain.
- **Confidential on-chain computation** is computation done inside circuits of the threshold FHE scheme with a fixed public key $pk$. We let smart contract programmers specify arbitrary logic inside FHE circuits.
- **User off-chain computation** captures computation done by users to construct transactions. They have access to user secrets and (possibly stale) contract states.
- **Zero-knowledge computation**, done via zk-SNARK system $\Pi$, pertains to both the off-chain prover component run by users and the on-chain verification component run by contracts.

We write pseudocode to describe all four forms of computation (for an example, see Figure 3), with the understanding that one programming framework can be designed to unify these four different execution environments.

**Attacks and mitigations.** We remark that a naive design is vulnerable to attacks where an attacker copies the confidential FHE state of a target contract to an attacking contract (with arbitrary logic) for decryption. We design a mechanism that achieves proper contract scope separation and prevents such attacks, utilizing simulation knowledge soundness of the zk-SNARK.

**Maximally privacy-preserving applications** (Section 5). Finally, to show case the expressiveness of our framework, we design designs for constant function market makers (CFMMs) and first-price sealed-bid auctions (FPSBAs) with maximal privacy guarantees.

By maximal privacy guarantees we mean that, additional to information that is required to be released for liveness of the application:

- External observers do not learn anything about confidential user inputs or confidential applications state.
- Users only learn binary information on whether their inputs were validly executed or not.

Additionally, for both applications, the confidential FHE computation contain only a couple of arithmetic operations on (e.g. 64-bit) integers, making it practical under the performance characteristics of existing FHE schemes on GPUs.

To keep the confidential FHE state small, we utilize existing techniques to do privacy-preserving token accounting. In particular, underlying both applications is a multi-asset shielded pool inspired by the Orchard design [41] of Zcash [8]. To have it interoperate with the on-chain FHE service, we modify the action circuit to declare net value change and asset type encrypted under the fixed chain FHE key $pk$.

**Privacy-preserving CFMMs.** Constant function market makers [1], a special case of automated market makers have seen tremendous adoption following the launch of Uniswap. We provide a simple design where users construct exact trades that are executed in sequence. Our construction has no additional information leakage on
trades or asset reserves beyond the programmably-determined re-
lease of spot price. Furthermore, since users construct exact trades,
they only learn a binary outcome.

Privacy-preserving FSBAs. Recently, there has been a surge in
auctions conducted on blockchains due to rising interest in non-
fungible tokens (NFTs). We ask if auctions can be conducted so
that non-winning bids are never revealed to anyone even after the
settlement of the auction. Such property allows asset reserves of
users to remain hidden and are extremely desirable for recurring
auctions. We provide a simple application design for first-price
sealed-bid auctions with such privacy guarantee.

2 PRELIMINARIES

In this section, we review the various building blocks including
distributed consensus via Byzantine fault tolerance (BFT), universal
and transparent zero-knowledge succinct non-interactive argu-
ment of knowledge (zk-SNARKs), threshold cryptography, as well as fully
homomorphic encryption (FHE).

Global setup. Throughout, we fix $n$ parties out of which $f$ are ad-
versarial. We assume that there is a global setup phase Setup where
a public parameter $pp$ and private keys for each party $sk_1, \ldots, sk_n$ are
generated. All statements involving $pp$, $sk_1, \ldots, sk_n$ are quanti-
fied over the randomness of such a setup process. We write $[[sk_i]]_S$ where $S \subseteq \{1, \ldots, n\}$ to denote private inputs or outputs for proto-
cols, i.e. each party $i \in S$ takes private input or obtains private out-
put $sk_i$. Looking ahead, Setup shall include any setup required for
sub-protocols (e.g. distributed key generation) and cryptographic primiti-
ves (e.g. zk-SNARKs). We use $\lambda$ to denote the security pa-
rameter and $\mathsf{poly}(\cdot, \cdot)$ to denote the classes of polynomial and negligible functions
respectively.

2.1 Consensus and State Machine Replication

A state machine $M$ consists of an efficient state update function
$M.\text{Update}$, with function signature $M.\text{Update}(st, in) \to st'$, where $st \in \{0, 1\}^*$, $in \in M.\text{IN}$, and $st' \in \{0, 1\}^* \cup \perp$. For any state $st$, we say
that $st \to st'$ is a valid transition upon input in if $M.\text{Update}(st, in) = st' \neq \perp$. A chain $C$ of length $n$ for $M$ is a sequence $C = (st_0 = e, in_1, st_1, \ldots, in_n, st_n)$, such that $st_i \to st_{i+1}$ is a valid transition
under input $in_i$ for all $i = 1, \ldots, n$. We say that $C \preceq C'$ if both $C$ and $C'$ are valid chains and $C$ is a prefix of $C'$.

Blockchains are state machines replicated via distributed consen-
sus mechanisms. We focus on Byzantine consensus mechanisms for
"streamlined blockchains" with a fixed set of participants per round,
examples of such include Tendermint [14], Casper [16], HotStuff
[51], Internet Computer Consensus [17], and Streamlet [18].

Streamlined BFT consensus protocols operate in a synchronous
or partially synchronous setting and proceeds in rounds. Nodes send
and receive messages to each other in each round via authenticated
channels. Preceding each round $r$, the consensus algorithm at node $i$
takes input a state machine input $in_i^j$. At the end of each round $r$,
the consensus algorithm at node $i$ outputs a finalized chain $C_{r}^i$
for machine $M$.

Safety asks that any finalized chains from two honest nodes must
agree, meaning that either $C^i_r \preceq C^j_{r'}$ or $C^j_{r'} \preceq C^i_r$ for any $r$ and $r'$
and honest nodes $i$ and $j$. Liveness asks that if one honest node
finalizes a chain $C^i_r$ at round $r$, then any other honest node must
finalize a chain extending $C^i_r$ by some finite round $r' \geq r$.

2.2 Non-interactive Zero-knowledge

We require a universal or transparent 2k-SNARK, examples of which
include Sonic [44], Marlin [20], and various variants of Plonk [34].

We follow the definition framework of [20] for preprocessing
SNARKs and adopt a game-based definition for NIZKs following [7].
An indexed relation $R$ consists of tuples of the form $(C, x, w)$, where $C$ is the
indicated machine $x$ is the instance, and $w$ the witness. Throughout,
we fix an indexed relation $\mathbb{R}_{p,n}$ where each index $C$ is the
encoding of an arithmetic circuit over prime field $\mathbb{F}_p$ of size at most $N$. Relation $\mathbb{R}_{p,n}$ is defined as $(C, x, w) \in \mathbb{R}_{p,n}$ if $C(x, w) \neq 0$
and $\left| C \right| \leq N$.

A universal non-interactive proof system II for $\mathbb{R}_{p}$ consists of algo-
rithms $\Pi.\text{Setup}$, $\Pi.\text{Compile}$, $\Pi.P$, and $\Pi.V$. We require the circuit
compiler $\Pi.\text{Compile}$ also known as the indexer) to be deterministic.
We say that $\Pi$ is succinctness if for any $pp \to \Pi.\text{Setup}(\lambda)$, the running
time of $V(pp, x, \lambda)$ is $\mathsf{poly}(\lambda, |x|)$. When succinctness is not
emphasized, and particularly inside security definitions and proofs,
we opt to omit writing of Compile and write $P(pp, (C, x), w)$ and
$V(pp, (C, x), \pi)$ to denote $P(pp, pk_C, x, w)$ and $V(pp, vk_C, x, \pi)$ re-
spectively, where $(pk_C, vk_C) = \Pi.\text{Compile}(pp, C)$. We consider the
usual definition of completeness, zero-knowledge, and simulation
knowledge soundness for the induced (non-universal and non-
succinct) proof system on relation $\mathbb{R}_{p}$ consisting of tuples of the
form $((C, x), w)$. Completeness. Perfect completeness says that for any adversary $A$,
the following game $G^{\text{cmp}}_{\Pi_A}$ outputs true with probability $1$.

Game $G^{\text{cmp}}_{\Pi_A}(\lambda)$

1. $pp \to \Pi.\text{Setup}(\lambda') : (x, w) \to \Pi.\text{Compile}(\lambda') ; \pi \to \Pi.P(pp, x, \lambda)$
2. $\Pi.V(pp, x, \pi)$

Zero-knowledge and simulation knowledge soundness. To define
zero-knowledge and simulation knowledge soundness, we require a
zero-knowledge simulator $\Pi.S$, its associated setup algorithm
$\Pi.S\text{Setup}$, as well as an extractor $\Pi.\text{Ext}$. Consider the zero-knowledge
game $G^{zk}_{\Pi_A}$ given below.

Game $G^{zk}_{\Pi_A}(\lambda)$

1. $b \leftarrow \{0, 1\}$
2. $(pp_b, \text{Id}) \leftarrow \Pi.S\text{Setup}
3. pp_{\pi_b} \leftarrow \Pi.P(pp_b, x, \lambda)$
4. $b \leftarrow A'_b(pp_{\pi_b})$
5. $\Pi.\text{Ext}(b' \mapsto \Pi.\text{Ext}(pp_{\pi_b}, x, \lambda))$
6. $\Pi.S\text{Setup}(pp_b, x, \lambda) \leftarrow \Pi.S\text{Setup}(pp_{\pi_b}, x, \lambda)$
7. $\Pi.\text{Ext}(b' \mapsto \Pi.\text{Ext}(pp_{\pi_b}, x, \lambda))$
8. $\Pi.\text{Ext}(b' \mapsto \Pi.\text{Ext}(pp_{\pi_b}, x, \lambda))$

We define the zero-knowledge advantage of an adversary $A$ to be
$\text{Adv}^{zk}_{\Pi_A}(\lambda) = 2 \cdot \mathsf{Pr}[G^{zk}_{\Pi_A}(\lambda) = 1] - 1$.

We move on to define simulation extractability [27, 38, 49].
Which is a strong notion requiring that the extractor to extract
valid witnesses from forged proofs for an adversary even if the ad-
versary has seen simulated proofs on possibly incorrect instances.
Formally, consider the game $G^{\text{ext}}_{\Pi_A}(\lambda)$ given below.
We define the simulation extractable (XT) advantage of \( \mathcal{A} \) against \( \Pi \) to be \( \text{Adv}^{\text{XT}}_{\Pi,\mathcal{A}}(\lambda) := \Pr[\mathcal{G}^\text{XT}_{\Pi,\mathcal{A}}(\lambda)] \). As usual, we say that \( \Pi \) is zero-knowledge and simulation knowledge sound if the corresponding advantages are negligible in \( \lambda \) for polynomial-time adversaries. We remark that Plonk and Sonic are known to simulation extractable advantages are negligible in (aka knowledge sound) [42] in the random oracle model.

### 2.3 Threshold FHE

Threshold decryption for a public-key encryption scheme allows splitting of the master decryption key into \( n \) key shares so that decryption can be done given any \( t \) shares without reconstructing the master key. Moreover, the reconstruction of the master key is not possible given less than \( t \) shares. This makes \( t \)-out-of-\( n \) threshold cryptography a natural candidate for distributed systems with similar adversarial assumptions, such as proof-of-stake (PoS) and Byzantine fault-tolerant (BFT) consensus systems.

It is well-known that public-key additively homomorphic encryption schemes such as ElGamal [31] and Paillier [47] can support threshold decryption [28] with Shamir [50] secret-shared keys. Fully homomorphic encryption (FHE) schemes can also be constructed to support threshold decryption [10, 11].

We fix a set of complete boolean gates \( \mathcal{G} \) where each \( g \in \mathcal{G} \) is a function of the form \( \{0, 1\}^f \rightarrow \{0, 1\} \).

**Threshold FHE** [10]. A threshold FHE scheme, supporting \( \mathcal{G} \), consists of algorithms\footnote{For simplicity this could be the singleton set containing the xnor gate. For efficiency, we may have many fan-in 2 and 3 gates such as and, or, and nand.} of probabilistic polynomial-time algorithms Kg, Enc, Eval, PartDec, FinDec.

- **Key generation.** Kg\((1^t,n) \rightarrow (pk, sk_1, \ldots, sk_n)\), upon input the unary security parameter, threshold \( t \), and size \( n \), key generation returns the public key \( pk \) and \( n \) secret keys.
- **Encryption.** Enc\((pk, m) \rightarrow c\) for any message \( m \in \{0, 1\} \). Encryption of a bit \( m \) takes additional input only on the public key \( pk \).
- **Homomorphic evaluation.** For any gate \( g \in \mathcal{G} \). Suppose \( c_1, \ldots, c_f \) are ciphertexts each encrypting one bit, then \( \text{Eval}_{pk}(f, (c_1, \ldots, c_f)) \) is a ciphertext.
- **Decryption of a ciphertext\( c \), denoted Dec\(([sk_i])_{i=1,...,n}, c\), is distributed protocol with two algorithms PartDec and FinDec. First, each \( i \)-th party computes partial decryption as \( d_i \leftarrow \text{PartDec}(sk_i, c) \). Combining at least \( t \) honest partial decryptions, any party can obtain the final decryption via \( m \leftarrow \text{FinDec}(pk, \{[i, d_i]\}_{i \in \mathcal{S}}) \).

We refer to [11, Section 5] for a formal treatment of compactness, correctness and security. Roughly, compactness says that decryption should be efficient regardless of the depth of the computation done on the ciphertexts. Correctness says that any homomorphic evaluations on honestly generated ciphertexts should yield the correct decryption. Standard (IND-CPA) security asks that an encryption of 0 to be indistinguishable from an encryption of 1.

Simulation-based threshold security asks that as long as less than threshold \( t \) number of parties are corrupt, decryption shares are simulatable without access to real secret keys.

**Verifiable partial decryption.** We require a strengthening of threshold decryption where each partial decryption share can be publicly verified. Specifically, we require an algorithm PartVerify, which decides the validity of a partial decryption \( d \) as a decision bit PartVerify\((pk, i, d_i, c)\). We say that PartVerify satisfies correctness if for \( d_i \leftarrow \text{PartDec}(sk_i, c) \), we have PartVerify\((pk, i, d_i, c) = 1 \). Intuitively, security requires that without knowledge of \( sk_i \), an adversary cannot construct correct new decryption shares, even after observing partial decryption shares for adversarially chosen ciphertexts. We remark that, efficiency aside, verifiable partial decryption can be obtained generically via simulation extractable zk-SNARKs, where we additionally include a commitment of each secret key \( sk_i \) inside the public key \( pk \). Specifically, we can consider PartDec’ and PartVerify, which utilizes a zk-SNARK for a circuit \( C \) computing a commitment and PartDec, as follows.

**Circuit** \( C(m, c, d, r_{\text{PartDec}}) \):
1. Assert \((cm = \text{Com}(sk, r_{\text{PartDec}}) \) and \( d = \text{PartDec}(sk, c, r_{\text{PartDec}}) \))

**Algorithm PartDec’** \((pk, sk_i, c, r_{\text{PartDec}}) \):
1. \( d \leftarrow \text{PartDec}(pk, sk_i, c, r_{\text{PartDec}}) \)

**Algorithm PartVerify**\((pk, i, (d, c), c)\):
5. \( \text{Ass} \text{ert} II.V(pp, C, (pk, cm_i, c, d), \pi) \)

**Shamir secret sharing.** For any prime field \( \mathbb{F}_p \), the \( t \)-out-of-\( n \) Shamir secret sharing scheme [50] over \( \mathbb{F}_p \) shares a secret \( s \in \mathbb{F}_p \) into \( n \) shares \( p(1), \ldots, p(n) \), where \( p \) is a uniform randomly sampled degree \( t + 1 \) polynomial such that \( p(0) = s \).

Based on the learning-with-errors (LWE) problem, a special class of FHE schemes [10, Definition 3.9], covering BGV [13] GSW [36] and FHEW [30] families, can be made compatible with Shamir secret keys. We summarize the modifications required on these schemes in Appendix A.

Note that naively as shown here, we require a trusted third party to run the key generation algorithm Kg and distribute key shares to relevant parties. This is undesirable for blockchain systems due to centralization. We assume the existence of a trusted third party in this work for simplicity but point out generic solutions to generate and manage Shamir secret shares. We do not formally specify or prove secure the composed systems here.

### 2.4 Threshold Key Management

Blockchain protocols deal with heterogeneous parties that may join and leave the protocol at arbitrary times. We are particularly interested in the \( t \)-out-of-\( n \) Shamir secret-sharing scheme [50], for which there are known distributed key generation (DKG) [26, 40] and dynamic proactive secret sharing (DPSS) [37, 40, 45] schemes. Furthermore, for large consensus sets where running DPSS over all nodes is not feasible, secrets can be kept and passed between small committees [9, 32, 35].

**Distributed key generation.** To generate Shamir secret shares without relying on a trusted party, a distributed key generation (DKG) protocol [26, 40] can be used. Concretely, the execution of a DKG protocol DKG\((\lambda, n, t) \leftarrow \{[sk_i]\}_{i=1,...,n}\) returns private outputs...
skᵢ to each party i, which form a t-out-of-n Shamir secret sharing of some master secret that is never reconstructed.

**Dynamic proactive secret sharing.** To support a dynamic consensus sets where nodes are joining and leaving, dynamic proactive secret sharing (DPSS) [37, 40, 45] can be utilized. Concretely, the execution of a DPSS protocol DPSS([[[skᵢ]ᵢ]₀]₀) requires at least t honest participants with their respective Shamir secret shares skᵢ and returns refreshed Shamir secret shares skᵢ for each participant i, while keeping the same master secret.

For the rest of the paper, we fix a threshold FHE scheme FHE with Shamir secrets where Kg is replaced by a compatible Shamir DKG protocol.

### 3 Replication of State Machines With Threshold Release

Blockchains are state machines replicated via distributed consensus protocols. While the state machine model is general, it only captures efficiently computable transitions, as the state machine is executed independently by each consensus node. In particular, the model does not capture transitions that require interaction between consensus to evaluate. One example of such blockchain design is threshold decryption. For example, both Ferveo [46] and Penumbral [48] modify the underlying consensus algorithm to support threshold decryption. In these protocols, the set of nodes will run a threshold decryption protocol to decrypt a pre-specified part of the state. The state machine then proceeds by taking input of the result of threshold decryption.

In this section, we provide a general definition of state machines with threshold release protocols and discuss how existing consensus protocols can be modified to replicate these state machines.

#### 3.1 State Machines and Release Functions

**State release function.** Let M be a state machine. A release function for M is a function Release : {0, 1}⁺ → {0, 1}⁺ which takes input a state st and outputs the state release information Release(st). Looking ahead, the release function is not assumed to be efficiently computable as a function but can be evaluated efficiently via an external protocol.

To model useful applications, it is often paramount for the release information to be fed back into the state machine as input. To model this, we consider the input space IN as {0, 1}⁺ × (N×{0, 1}⁺)⁺, where the second component encodes an arbitrary sequence of release information from previous states.

For a particular sequence of inputs in₁, . . . and correctly computed valid states s₀, . . . , we say that sᵢ is released with delay t if inᵢ₋₁_d contains (i, Release(sᵢ)). We let M.Update encode the validity conditions on the necessary delay. For example, M.Update could reject inputs inᵢ₋₁ unless it contains a release of the previous state Release(stᵢ₋₁) is provided.

#### 3.2 Replicating Threshold-release Machines

**Threshold release protocol.** We focus on a release function with threshold release protocols. A (t, n)-threshold protocol ThRelease to compute Release consists of four efficient algorithms Setup, PartRelease, Combine, and PartVerify. Consider the following (strong) correctness game against any polynomial-time adversary A.

**Game G^thec^{ThRelease, A}(λ).**

1. (pp, sk₁, . . . , skₙ) ← Setup(1^λ)
2. S ← A(pp, Setup); Assert |S| < t
3. (x, y) ← A^{ Eval}(skᵢ, 1:c)
4. y' ← Combine(pp, PartRelease(skᵢ, x)₁:c–́)
5. Return |x ∈ Q and (y = y')|
6. Eval(i, x): 
   - Q ← u; Return PartRelease(skᵢ, x)

We say ThRelease is secure if Pr[G^thec^{ThRelease, A}(λ) ⇒ true] is negligible in λ.

Let M be a state machine with a (t, n)-threshold release protocol for Release. We describe how existing consensus protocol on n nodes can be modified to replicate M with the following property:

- **Release liveness.** If state sᵢ is finalized, then a valid release (Release(sᵢ), D) appears in the input inᵢ of some finalized state sᵢ for j > i.
- **Release safety.** If a valid release (Release(sᵢ), D) appears in the input inᵢ of any state sᵢ (not necessarily finalized), then it must be that sᵢ is finalized (for some honest node).

We remark that the above two properties can be achieved by modifying FHE protocols as long as f+1 ≤ t ≤ 2f+1 as follows:

1. In any round, if a state st becomes finalized for node i, then it will release and broadcast its release share PartReleaseᵢ(st).
2. If a node receives some set D of t release shares that forms a valid release witness for some st, then it computes Release(st) = Combinepp(D).
3. In any round, if the leader knows a finalized state st and its release information Release(st) with release witness D but has not observed Release(st) in any proposed blocks, it will include (Release(st), D) in the newly proposed state machine input in.

We claim that with the above modifications, any FHE protocol satisfying safety and liveness additionally satisfies release safety and release liveness.

**Proposition 3.1.** If a FHE protocol satisfies liveness, then the modified FHE protocol satisfy release liveness assuming partially synchronous network.
Proposition 3.2. If a BFT protocol satisfies safety and ThRelease is a secure threshold release protocol, then the modified BFT protocol satisfy release safety.

Proof. Suppose a valid release (Release(st), D = { (i, di) }) appears in some input inj. We know that with overwhelming probability, due to the security of ThRelease, that all di values are computed via PartRelease(sk, st). This means that at least one (t − f ≥ 1) honest nodes have considered st finalized. □

4 PRIVACY-ENHANCING SMART CONTRACT

In this section, we describe an architecture for expressively programming of state machines with threshold release.

Overview of a smart contract platform. We base our architecture off of standard smart contract platforms such as Ethereum, Solana, and Dfinity. Smart contracts are stateful deterministic programs that run on a blockchain. We assume that smart contracts can invoke "public" functions of other contracts and that such calls are blocking. We assume that each execution of a smart contract is initiated by an end-user via a transaction. Errors encountered during the execution of smart contracts revert the entire transaction. Each contract has a distinct owner and a contract identifier which is programmatically accessible via the variable this.

Compared to smart contract platforms such as Ethereum, our architecture adds the ability to programmatically invoke threshold decryption of an FHE scheme. As a result, we have four types of computation of interest.

Transparent on-chain computation. This is the bread and butter of smart contract platforms. Transparent on-chain computation is replicated among the set of all consensus nodes and hence public and transparent. In pseudocode, they are denoted as either public or private contract functions depending on standard programming access control idioms. Public refers to the fact that it can be invoked by external contracts. Private refers to the fact that it can only be invoked within the contract itself.

User off-chain computation. Often, users need to provide structured inputs to on-chain smart contracts. This is especially true for users that need to provide zero-knowledge proofs. We denote user computation as User functions. User functions are allowed access to contract storage as well as user-specific secret information such as keys. Note that user off-chain computation are never deployed to chain.

Zero-knowledge off-chain computation. With the help of a universal or transparent zk-SNARK II, we can express any validity check as a circuit. In pseudocode, they are denoted as ZK circuits. Zero-knowledge circuits only return a boolean value and do not have access to contract storage. We assume that the for each ZK circuit C, the contract owner computes (pkC, vkC) ← II.Compile(C) before deploying the contract. In the same contract, C.verify(x; π) is a shorthand for II.V(vkC, (this, x), π). For any user function, C.prove(x; w) is a shorthand for II.P(pkC, (this, x), w). Here, we draw attention to the use of the contract identifier this as part of the instance. The reason is to properly scope separate proofs for one contract from another.

Confidential (fully homomorphic) on-chain computation. With the help of a threshold FHE scheme, we can express any computation on confidential states as FHE circuits. FHE circuits do not have access to contract storage. Inside any contract function and for any FHE circuit C of the contract, we write C.Eval(ein) as a shorthand for FHE.Evalpk(C, ein).

Threshold decryption. Additionally, at any point during a transparent on-chain computation, a call to threshold decryption can be made. The call is made with three parameters, some FHE ciphertexts encrypted to be decrypted, a variable x to store the decrypted value, and an asynchronous code block that is executed after the threshold decryption result is made available. We write this as "Async x ← ThDec(est)" followed by the code block. We allow state read and access in the asynchronous code block, with the understanding that the content of the contract state can change arbitrarily between the threshold decryption call and the execution of the asynchronous code block.

4.1 Safety of FHE application states

Since anyone can deploy a contract that invokes threshold decryption, an adversary can decrypt any previously observed FHE state and call threshold decryption on it. To prevent this attack, we employ the zk-SNARK to enforce "knowledge of secret" of any new FHE ciphertexts. Consider the following zero-knowledge circuit and contract function.

Contract FHEBase
Par Func VerifyFHEInput(eb):
1. Assert KnowledgeCheck.verify(eb; eb.π)
ZK CIRCUIT KnowledgeCheck(eb; b, r):
2. Assert (eb = FHE.Encpk(b; r))

Figure 1: Base contract for utilizing FHE state and threshold decryption.

We remark that each contract that utilizes the on-chain FHE functionality must use the FHEBase contract as a library. However, proofs constructed for KnowledgeCheck are not compatible across different contracts, as we assume the contract identifier this is inserted into the public instance, i.e. KnowledgeCheck.verify(eb; π) = II.V(vkKnowledgeCheck, (this, eb), π). By simulation extractability of II, proofs for different contract identifiers are properly separated. We require the state machine to ensure that:

- An input FHE ciphertext eb is marked valid only if it contains a valid knowledge proofs, i.e. VerifyFHEInput(eb) = true.
- Inputs to FHE.Evalpk(−) and ThDec must only contain valid FHE ciphertexts.
- Outputs of FHE.Evalpk are marked as valid.
The above can be done by exposing FHE.Evalpk and ThDec as special “system calls.”

We claim that if the state machine enforces the above validity rules, then trivial attacks on privacy cannot occur.

5 APPLICATIONS

In this section, we outline two applications programmed via Pesca framework—an constant function market makers (CFMMs) and first-price sealed-bid auctions (FPSBAs), both providing maximal privacy guarantees.

We chose these two applications to showcase the expressiveness of the framework due to their simplicity and limited shared state—both applications only maintain a small constant shared state regardless of how many users are interacting with the application. To achieve this property, we utilize techniques from Zerocash to do token accounting and usage. We utilize an on-chain threshold FHE service to compute over shared states of CFMMs and FPSBAs, which are of size independent of the number of users interacting with these applications. Connecting between them, we require that spends to declare inputs encrypted to the FHE service, which is done inside the zero-knowledge circuits for spending tokens (called Action circuits).

5.1 Shielded Assets

At the foundation of our privacy-preserving trading applications is a design of a multi-asset shielded pool supporting token usage that declares inputs to the on-chain FHE service. For this shielded pool, we take a design approach closely following the Orchard design of the Zcash protocol [41], which is the third iteration of the pool, we take a design approach closely following the Orchard design of the Zcash protocol [41], which is the third iteration of the framework–constant function market makers (CFMMs) and first-price sealed-bid auctions (FPSBAs), both providing maximal privacy guarantees.

The state of our smart contract consists of a Merkle tree of notes $\mathbf{MT}$ and a set of spent note nullifiers $\mathbf{NS}$. We assume a Merkle tree implementation supporting insertion and lookups via $\mathbf{MT}.\text{add}$ and $\mathbf{MT}[\text{path}]$. We also assume that the Merkle tree implementation exposes a list of periodically snapshots historical Merkle tree roots via $\mathbf{MT}.\text{RTS}$.

**Notes**. A note is a fundamental data unit encoding four values, the public key $\mathbf{pk}$ of the owner, the asset identifier $\mathbf{typ}$, a nonce $\mathbf{\rho}$, a value $\mathbf{val} \in V$, where $\mathbf{V}$ is a set of allowable values, e.g. $\mathbf{V} = \{0, \ldots, 2^{64} - 1\}$. We write $\mathbf{note} = (\mathbf{pk}, \mathbf{typ}, \mathbf{val}, \mathbf{\rho})$. Associated with each note are two algorithms $\text{NoteCommit}$ and $\text{DeriveNullifier}$. Algorithm $\text{NoteCommit}$ takes input a note and returns a note commitment $\mathbf{ncm}$. Algorithm $\text{DeriveNullifier}$ takes input a secret key $\mathbf{sk}$ and a note note to return a nullifier $\mathbf{nf}$.

**The Action circuit**. The action circuit declares a transaction that spends a note (via revealing a nullifier $\mathbf{nf}$) and creates a note (via revealing a note commitment $\mathbf{ncm}$). It also declares the two fields $\mathbf{tx.ev}$ and $\mathbf{tx.et}$, encrypting the net value change and asset type (of both notes) under the FHE public key $\mathbf{pk}$. We let the action circuit declare these values so that they can be passed to FHE circuits as inputs.

**Transaction creation**. The user function $\text{CreateTx}$ contains code that creates a transaction. It takes input a secret key $\mathbf{sk}$, some $\mathbf{actionInfo}$ containing path to a note $\mathbf{oldNote}$ owned by the user, a new $\mathbf{newNote}$ to be created, and encryption randomness $\mathbf{r_{val}}$ and $\mathbf{r_{typ}}$. It prepares a transaction $\mathbf{tx}$ containing fields $\mathbf{rt}$, $\mathbf{ncm}$, $\mathbf{nf}$, $\mathbf{ev}$, $\mathbf{et}$, and $\mathbf{\pi}$. The Merkle root $\mathbf{rt}$ of the transaction is set to the most recently available Merkle root $\mathbf{MT}.\mathbf{rt}$. The note commitment $\mathbf{ncm}$ is set to the commitment of the created $\mathbf{newNote}$. The nullifier $\mathbf{nf}$ is set to the derived nullifier of note to be spent $\mathbf{oldNote}$ using the user secret key $\mathbf{sk}$. The encrypted value $\mathbf{ev}$ is set to the net difference (a signed integer) between $\mathbf{oldNote.val}$ and $\mathbf{newNote.val}$. The encrypted type $\mathbf{et}$ is set to the encryption of the type that both notes must share. Finally, the validity proof $\mathbf{\pi}$ is computed by invoking the zk-SNARK prover for the $\text{Action}$ circuit.

**Transaction processing**. Unlike Zerocash, we do not balance of values when transactions are processed. Private (in terms of accessibility) contract function $\text{Process}$ first verifies that the spend encoded in a transaction $\mathbf{tx}$ is valid before processing it. Validity checking amounts to checking that if $\mathbf{tx}.\mathbf{nf} \neq \perp$ then $\mathbf{tx}.\mathbf{rt}$ must have been a historical Merkle root and that the $\text{Action}$ circuit proof is valid. Processing of a $\mathbf{tx}$ simply adds the new note commitment $\mathbf{ncm}$ to the Merkle tree $\mathbf{MT}$ and the spent note nullifier $\mathbf{nf}$ to $\mathbf{NS}$.

**The Balance circuit**. “Balance” refers to that two transactions $\mathbf{tx}$ and $\mathbf{tx’}$ declare the exact opposite value changes of the same asset type. In Zcash Orchard, balance is checked by utilizing the additive homomorphism of the commitment scheme. We use a ZK circuit to ensure balance here. We note that we could simplify the balance circuit by utilizing additionally another binding commitment scheme instead of FHE.Enc since full homomorphism is not required. But we opt to present the concept in an expressive manner without considering constant factor efficiency differences.

**Public type declaration**. We have chosen to hide the asset types by default for transactions to enable multi-asset confidential transfers. However, it shall be convenient for our later applications for transactions to (optionally) publicly declare the asset type. This is done via fields $\mathbf{tx.typ}$ and $\mathbf{tx.rtyp}$. We require the type declaration to be correct, meaning that $\mathbf{tx.et} = \text{FHE.Enc}_{\mathbf{pk}}(\mathbf{tx.typ}; \mathbf{tx.rtyp})$—a check that is checked implicitly for all incoming transactions.

**Relations with Decentralized Anonymous Payments (DAPs)**. A closely related primitive, first formalized in the work of Zerocash [8] is called Decentralized Anonymous Payments (DAPs). A DAP allows anonymous and confidential transactions modifying account balances in a verifiable manner. A ledger maintains a state that encodes the balances of users. To transact, users submit a transaction declaring certain properties, which is then processed by the ledger. For us, a transaction $\mathbf{tx}$ can declare any amounts of tokens minted (created) or burnt (destroyed), which the corresponding value declared via $\mathbf{tx.ev}$. With the Balance circuit, our construction do realize the properties of DAPs but we do not formalize this here since it is not the focus of our work and due to syntactical differences.
with mutation of batched trades, we can provably achieve differential privacy. The application state consists of three numbers $\mathbf{x}, \mathbf{y}, \mathbf{z}$ representing the overall asset reserves and integers interpreted as fixed-point rational numbers.

5.2 Trading: constant function market makers

Constant function market makers (CFMMs) have seen wide adoption on computing and storage constrained smart contract platforms such as Ethereum. Applications such as Curve and Uniswap, which are special cases of CFMMs, manage billions worth of assets and daily trades. CFMMs allow traders to trade against assets of liquidity providers (LPs) according to a fixed, simple rule. Mathematically, the application state consists of three numbers $(x, y, z) \in \mathbb{R}^3$, with $x, y$ representing the overall asset reserves and $z$ representing the representing pool tokens. In absence of fees, a trade of $(\delta_x, \delta_y) \in \mathbb{R}^2$ is valid if and only if

$$
\phi(x + \delta_x, y + \delta_y) \geq \phi(x, y),
$$

where $\phi$ is a function called the trading function. For simplicity, we focus on the widely studied and adopted constant-product function in this work, i.e., $\phi(x, y) = xy$. In applications, reserve and trading amounts are typically represented as unsigned or signed 64-bit integers interpreted as fixed-point rational numbers.

Trade privacy in CFMMs. Two main privacy metrics of CFMMs are anonymity and trade privacy [25]. Roughly, A CFMM has anonymity if observers cannot infer the identity of the traders and it has trade privacy if trade amounts are not revealed. On one hand, it is known how to protect the anonymity of the traders [23, 24] using zero-knowledge proofs. On the other hand, it is known that trade privacy cannot be achieved if exact spot prices are released for each trade [2]. There are solutions of using ElGamal to net trades in a batch before executing [48]. Here, we describe how trades can be executed sequentially with programmable price release, as an application in the PESCA framework. Together with random permutation of batched trades, we can provably achieve differential privacy of individual execution prices [22].

FHE trade circuit. We first describe the key ingredient in our construction—the FHE trade circuit. We utilize the threshold FHE functionality to decide the validity of an incoming trade, which is encoded in a bit $b$ (with 1 being valid and 0 otherwise) that is decrypted asynchronously via threshold decryption.

Contract CFM extends Token, FHEBase

is encoded in a bit $b$ (with 1 being valid and 0 otherwise) that is decrypted asynchronously via threshold decryption.

Contract function Trade. To interface with the FHE Trade circuit, three action transactions are required: $tx_{\text{fund}}, tx_{\text{refund}}$ and $tx_{\text{payout}}$. Transactions $tx_{\text{fund}}$ and $tx_{\text{refund}}$ should balance, meaning that they spend and refunds the exact same amount. Transactions $tx_{\text{fund}}$ and $tx_{\text{payout}}$ if processed together, reflect that the trade is valid. Transactions $tx_{\text{fund}}$ and $tx_{\text{refund}}$ if processed together, reflect that the trade is invalid. Exactly which two transactions get processed is determined from the output bit of the FHE Trade circuit.

Price release and liquidity provision. To enable trades, the spot price of the market must be periodically released. We encapsulate
this in a contract function and FHE circuit ReleasePrice given in Figure 4. When invoked, the function simply releases the amount of normalized balance per pool token of either asset. We assume that there are external mechanisms, e.g. timers, that periodically invoke the price release mechanism so that users and arbitrageurs can trade against the CFMM.

<table>
<thead>
<tr>
<th>Contract FPA extends Token, FHEBase</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>STATE variables:</strong></td>
</tr>
<tr>
<td>1. public est // List of running auction states</td>
</tr>
<tr>
<td>2. public item // List of auction items</td>
</tr>
<tr>
<td>3. public refund // List of bid refunds for each auction item</td>
</tr>
<tr>
<td>4. public payout // List of payout transactions for each auction item</td>
</tr>
<tr>
<td><strong>Func SetupAuction(txfund, txrefund, π):</strong></td>
</tr>
<tr>
<td>5. Balance.verify((txfund, txrefund, π) : this.Process(txfund))</td>
</tr>
<tr>
<td>6. i ← [this.est]; this.item[i] ← txfund</td>
</tr>
<tr>
<td>7. this.payout[i] ← txrefund</td>
</tr>
<tr>
<td>8. this.est[i] ← est; Return i</td>
</tr>
<tr>
<td><strong>FHE Circuit</strong></td>
</tr>
<tr>
<td>1. If (q &gt; p) then return (q, j) Else return (p, k)</td>
</tr>
<tr>
<td>2. Func ProcessBid(i, txfund, txrefund, tfund, tpayout, npayout):</td>
</tr>
<tr>
<td>3. Assert (txfund : typ) // Some fixed currency type</td>
</tr>
<tr>
<td>4. Assert Balance((txfund, txrefund, tfund))</td>
</tr>
<tr>
<td>5. Assert Balance((txfund[i], npayout))</td>
</tr>
<tr>
<td>6. Assert (txrefund[i] = (π, tfund, tfund, π))</td>
</tr>
<tr>
<td>7. ProcessBid(tfund[i], npayout[i]) ; Return (this.payout[i])</td>
</tr>
<tr>
<td><strong>Func FinalizeAuction(i):</strong></td>
</tr>
<tr>
<td>1. (, ej) ← this.est[i]</td>
</tr>
<tr>
<td>2. Async j ← ThDec(ej)</td>
</tr>
<tr>
<td>3. this.Process(this.payout[i][j])</td>
</tr>
</tbody>
</table>

**Figure 4: Contract design for privacy-preserving sealed-bid first-price auction.**

Providing liquidity. Besides trading, a CFMM requires the ability for users to deposit liquidity. The design here follows the same outline as the case for trade. The FHE circuit Enter takes an additional input z, which encodes the number of pool tokens minted or burnt. We omit further details here as the pseudocode given in Figure 4 is self-explanatory. We remark that for users to be able to construct valid deposits. The user must know the exact spot trading price, which is only possible after a price release. To make this possible, the CFMM contract can block trades at a predetermined time for a set amount of time and release the up-to-date spot price for liquidity provision.

### 5.3 Sealed-bid auctions

**Overview of contract.** The on-chain application act as the auctioneer, accepting encrypted (to chain FHE public key pk) bids from users. The contract is capable of keeping track of arbitrary number of concurrently running auctions, each with some index i. For each auction i, the application state est[i] consist of a tuple of (unsigned) integers (p, j), each initialized to 0. Integer p encodes the current max bid and j denotes the index of the max bidder. Here, for simplicity, we assume that all bids are given in some fixed token type Typ.

In a sealed-bid auction, each bidder commits to a price p_j which is not revealed to any other user. The winner is defined as the party j with the largest bid p_j, and the realized price is typically some function f of the first and second price.

We are interested in sealed-bid auctions with no user-to-user interaction and minimal information leakage. In particular, we want the bids of users to stay sealed even after the conclusion of the auction. This is potentially important for multi-round auctions, as asset reserves of bidders are not revealed.

**Setting up an auction.** To set up an auction, the item hold simply constructs a funding transaction txfund spending the auction item, a refund transaction txrefund which refunds the auction item (default outcome of auction), a balance proof π, an initial auction state est, encoding the minimum bid price, and a state initialization proof π'. We assume that the information about the auction item, namely its type, is released in some channel so that interested users can construct valid bidding transactions.

**Accumulate FHE circuit.** We design a minimal FHE circuit to support bid processing. Note that the Accum circuit is parameterized by an integer i denoting the incoming bid index. The circuit simply compares the current winning price to an incoming bid and updates the winner index accordingly.
Bid generation. To create a bid, a user simply generates a funding transaction \( t_{x_{\text{fund}}} \), associated refund transaction \( t_{x_{\text{refund}}} \), and balance proof \( \pi \).

Finalizing an auction. To finalize an auction with index \( j \), the contract first threshold decrypts the part of the state containing the winner index \( j \). It will process refund transactions of all losing bids and the payout transaction of the winning bidder. Note that since we have previously ensured balance, no additional balance checks are required in this step.

Information leakage analysis. Note that only the `FinalizeAuction` function release information about the confidential state, and only the winner index is revealed. This means that each bidder can only learn whether their bid was a winning bid or not.

Efficiency consideration. Our FHE circuits require only a couple arithmetic operations on fixed-point numbers. Unfortunately, the most efficient bootstrapped FHE schemes, such as FHEW [30] and TFHE [21], work over binary values. We note that to implement \( n \)-bit arithmetic operations using binary gates we need to account for blowup factors of \( O(n) \) for addition and \( m \times n, O(n^2) \) for multiplication and division.

6 CONCLUSION

In this work, we propose PESCA, a smart contract architecture where privacy-preserving applications can be built with its security properties solely reliant on the Byzantine assumptions of consensus nodes.

We formalize the notion of state machine augmented with threshold release protocols and study how existing BFT protocols can be adopted to replicate them. We instantiate a blockchain supporting programmable threshold decryption using threshold FHE amongst other widely-studied building blocks. We propose a smart contract architecture on top of such a blockchain where applications can be engineered expressively. We provide two example applications realizing CFMMs and FPSBAs, both with maximal privacy guarantees.

We identify the performance of threshold FHE schemes support Shamir secret keys as a key performance bottleneck. We think that with more research efforts on such types of FHE schemes, as well as more engineering efforts on programming and compiler frameworks for FHE and universal zk-SNARKs, the PESCA blueprint can become truly viable to support real-world applications.

REFERENCES


A SIZE OF MODULUS FOR THRESHOLD FHE

Fix a finite field $\mathbb{F}_p$. Shamir out-of-n secret sharing takes input
a master secret $sk \in \mathbb{F}_p$ and outputs $n$ secret shares $sk_1 = P(1), \ldots, sk_n = P(n)$ where $P \in \mathbb{F}_p[x]$ is a random polynomial of degree $t$ with the restriction that $P(0) = sk$. Given any subset $S$ of key shares with $|S| = t$, a reconstruction of any other share $P(j)$, including the master secret with $P(0)$, can be obtained as a linear combination of shares from $S$:

$$P(j) = \sum_{i \in S} \lambda_i^j \cdot P(i),$$  \hspace{1cm} (2)$$

where $\lambda_i^j$ are the Lagrange coefficients. Threshold decryption is obtained by observing that reconstruction and decryption are both linear and "commute", i.e. reconstruction can be done over partial decryptions.

Unlike ElGamal or Paillier, encryption schemes based on the learning-with-error (LWE) problem do not directly support threshold decryption naively as described above, as applying the Lagrange coefficients directly to LWE ciphertexts completely removes the information encoded on the message encrypted. The workaround proposed by [10, 11] involves computing $\sum_{i \in S} \lambda_i^j$ as a rational number instead of as a field element. During reconstruction, a large multiplier $(nt)^2$ is applied to both sides of (2) to clear out the denominators. As a consequence, the modulus $q$ needs to be "blown up" by a factor of $(nt)^3$ compared to that of non-threshold schemes. This method applies to a special class of FHE schemes [10, Definition 3.9] covering BGV [13] GSW [36] and FHEW [30] families.


